International Baccalaureate: Extended Essay

Prediction and investigation of the time required to completely drain a cylindrical tank filled with water

Research Question: What is the relationship between height of water in a tank and time needed to completely drain the tank

Physics

Contents

1.Essay Overview

Fluid dynamics is a complex and difficult area of physics; hence it is lightly covered even in the engineering option of IB physics. However in reality, even the seemingly simple phenomenon of water draining from a cylindrical tank contains complex features of the actual behaviour of fluids. As the IB only deals with ideal fluids and basics of fluid dynamics, I thought it would be interesting to analyse this simplified phenomenon in depth, modelling its actual behaviours without the assumptions of ideal fluids.

This essay is an attempt to answer the question, "**What is the relationship between the height of the water and the time needed to drain all the water in a tank?**". The main objective of the essay is to create a theoretical model from a modified version of Torricelli's law to answer the question and verify it with experimental data collected from a designed experiment. To validate the model's answer to the question, each modification made to the model must be individually verified by empirical data.

The first step for modification will be made by investigating the relationship between the length of a jet of water escaping from the tank's hole and the height of the water. The reason for this is because I predicted that the assumption of ignoring surface forces and viscous forces for ideal fluids is invalid and the height of the water will not stop at zero when the tank draining finishes. This investigation will introduce a correction factor, derived from the energy losses due to the forces mentioned above.

The second step is about investigating how turbulence affected the phenomenon. This step is also necessary as another invalid assumption of ideal fluids is that the flow of fluid is steady and without turbulence. However, all fluids experience turbulence and random bursts due to the fluctuations in the fluid's flow. Normally the amount of deviation they cause are negligible, but as the height decreases, the short random bursts get significantly noticeable and are superimposed onto the flow of water, affecting the total time it takes to drain the tank. This phenomenon is too complex to analyse. Hence, we will attempt to find a hypothetical drain time extrapolated from stable points of data to exclude the points that were affected by the bursts. The extrapolated value can then be compared to the theoretical prediction to justify that the modification made to the model was correct.

These steps will lead to the final investigation of finding each empirical time needed to drain the tank for all measured heights and comparing it with the predicted values from the theory, hence concluding whether the theory is correct.

2.Background Knowledge

Torricelli's Law and Bernoulli's Equation

Torricelli's law is a theory that predicts a relationship between the height of water above an opening and the velocity of the jet of water escaping the hole. When h=height of water from the hole, V_j=velocity of jet of water, and g=gravitational acceleration, Torricelli's law predicts,

$$V_j^2 = 2gh \qquad - 2.1$$

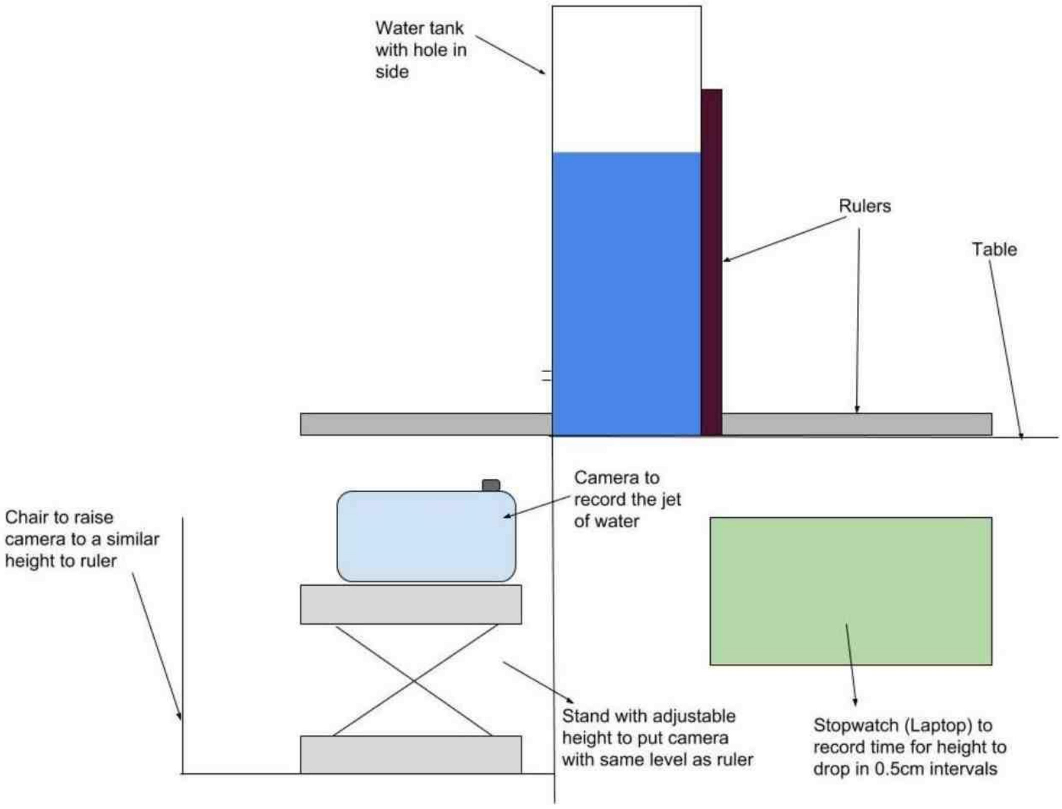
This law is also just one specific scenario within the general movements fluids can have. The general law for fluids can be described by Bernoulli's equation, which is derived from the energy conservation at all different points within a fluid. When p=pressure, ρ=density, V=velocity of the fluid, h= height of fluid from a datum, g=gravitational acceleration, Bernoulli's equation states

$$p_1 + \frac{1}{2}\rho_1 V_1^2 + \rho_1 g h_1 = \text{constant at any other point within the fluid} \qquad - 2.2$$

3.Experiment Setup

The experiment is designed to study the jet of water escaping the tank and the changes in the water inside the tank. *Diagram 1* represents the apparatus and setup of the experiment (refer to *Appendix 1* for a clearer 3-dimensional picture of the setup as it is difficult to represent with a 2-dimensional diagram)

Diagram 1: Apparatus and Experiment Setup



A hole with diameter 0.5cm is created in the tank with a sharp screwdriver. The radius of the cylindrical section of the tank is 4.3cm. Both were measured with ±0.1cm uncertainty.

There are essential steps to be made with caution during the setup to ensure accurate measurements. First, it is important to place the reading on the meter stick exactly at the edge of the table since that point is the reference point as 0. Likewise, the tank must be placed at the edge of the table, allowing the jet of water to properly start at the edge, i.e. 0 point on the metre stick. The camera must also be placed perpendicular to and at the same level as the ruler.

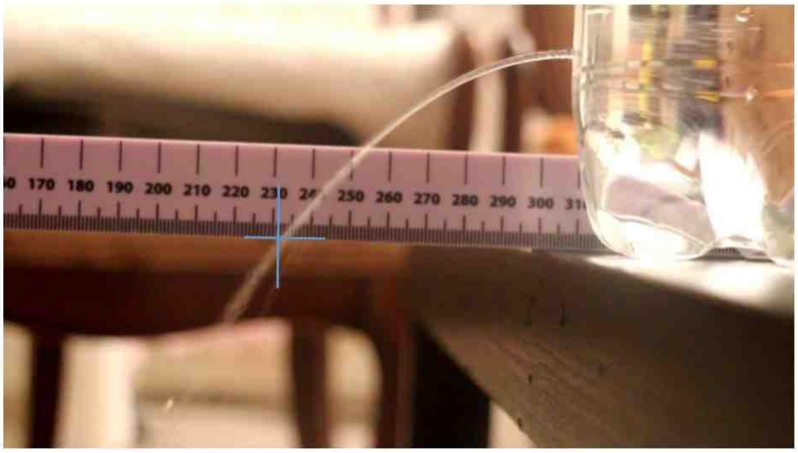
4.Experiment Procedure and Collected Data

Collection: 3 types of raw data will be collected in this experiment, length of jet (L), height of water from hole (h), and time elapsed (t).

First, a test run is made to find the height at which the drainage stops, deciding how to choose the heights so the last measurement matches the end of the draining in 0.5cm intervals. Then, the tank is filled above the desired initial height (14.0cm) and left until the water is at rest. Then the hole is uncovered and as the height of the water reaches the desired height, the camera is started to record the jet of water as shown in *Diagram 1*. Simultaneously, a stopwatch is also started. As the tank drains, the time taken for height to drop 0.5cm intervals is manually recorded with the stopwatch until the end. Care must be taken to measure the height at the meniscus.

After the experiment, the length of the jet at each respective height (each differing by 0.5cm) is measured by pausing the recorded video of the jet at the corresponding time it took to reach each height. It is also important to measure the length by using the same point of the jet, in this case, the further end of the jet which is shown in *Picture 1* as the intersection of the 2 lines. This complete procedure is repeated 3 times to obtain 3 separate values for each height to reduce random errors.

Picture 1: Point of measurement for jet of water



A sample of the 3 raw data values is shown below in *Table 1*, (full data table shown in *Appendix 2*)

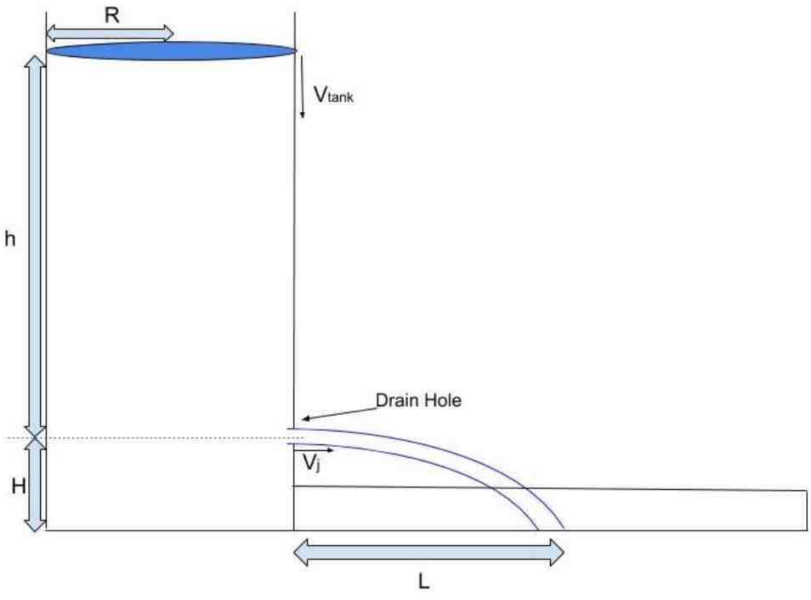
Table 1: Sample of recorded raw data

Trials	Height (cm) ±0.05	Time (sec) ±0.01	Length (cm) ±0.05
1	14.0	0.00	12.7
2		0.00	12.8
3		0.00	12.9
1	13.7	7.06	12.5
2		6.06	12.6
3		5.63	12.7
1	13.2	17.05	12.3
2		16.43	12.4
3		15.63	12.5

5.Definition of Symbols

There are many variables used during the measurements/calculations/proofs in this essay, requiring the use of shortened symbols. *Diagram 2* will present the basic variables to be used in this essay.

Diagram 2: Shortened symbols and their definitions



H: height from ruler to hole

h: height from hole to water

L: length of jet of water at ruler

R: radius of cylindrical tank

r: radius of drain hole

V_j: velocity of jet

V_T: velocity of water in tank

6.Correction of Torricelli's Law

Torricelli's law assumes that the water in the tank is steady and neglects the downwards velocity of the water. Here this assumption will not be made to derive a more accurate version of the law. To do this, the system balance formulation, a derivation of the mass conservation principle, is applied to both the drain hole and the surface of the water in the tank. It states that the volume of fluid escaping via the drain hole must equal the change in volume of the water in the tank (Recktenwald 4). Considering that the change in the tank is a negative value compared to the volume of fluid escaping, the equation becomes

$$\pi R^2 * V_T = -\pi r^2 * V_j \qquad - 6.1$$

from this we can isolate V_T, rewriting it as *dh/dt*

$$V_T = \frac{dh}{dt} = -\frac{r^2}{R^2} * V_j \qquad - 6.2$$

Then, we use the Bernoulli equation (refer to **2. Background Knowledge**) on both the drain hole (defined as point 1) and surface of the water in the tank (defined as point 2), giving

$$P_1 + \frac{1}{2}\rho_1 V_j^2 + \rho_1 g h_1 = P_2 + \frac{1}{2}\rho_2 \left(\frac{dh_2}{dt}\right)^2 + \rho_2 g h_2 \qquad - 6.3$$

However, the pressure at point 1 and 2 are both equal as the atmospheric pressure. Also, the height of point 1 is 0 as point 1 was defined as the datum. Finally, the density of water is equal at both points, and h_2 is the same as the h defined in **Section 5**. Hence the equation can be simplified as

$$V_j^2 = \left(\frac{dh}{dt}\right)^2 + 2gh \qquad - 6.4$$

Here we can substitute equation 6.2, giving

$$V_j^2 = \frac{r^4}{R^4} V_j^2 + 2gh \qquad - 6.5$$

And rearranging to get

$$V_j^2\left(1-\frac{r^4}{R^4}\right)=2gh \qquad -6.6$$

Here we can see Torricelli's law was missing a factor of $\left(1-\frac{r^4}{R^4}\right)$ multiplied to V_j^2. However, when calculating with the parameters of the experiment, the deviation from the original law (equation 2.1) is in the power of 10^{-5}, smaller than our precision in calculating the velocity (10^{-2}). Therefore, it is valid to ignore the missing factor and continue using the original Torricelli's law and consider the tank drainage flow as a "quasi-steady state of Bernoulli's equation" (Recktenwald 5). Refer to *Appendix 3* for detailed calculations

7.Determining the Effective Height

Torricelli's and Bernoulli's equations take the assumption that all fluids are ideal fluids. One of the properties of ideal fluids is that there are no energy losses through surface forces, which is not true in the real world. This section investigates the effects of surface tension and adhesion force and refines our model to describe the tank drain phenomenon as it is.

It can be predicted that mainly due to the two forces mentioned before, the level of the water will not be able to reach the hole. This is because the pressure due to the weight of the water acting at the drain hole will eventually balance with the surface tension and adhesion forces, preventing the water from escaping the hole. Empirical data will be used to find the height at which L reaches 0 and the relationship between L and h.

7.1. Theoretical Relationship between L and h

The derivation of an expression for L requires the equations for motion (a.k.a. SUVAT equations). We can work out the time it takes for the water to escape and reach the ruler by looking at the vertical component of the jet's motion. The motion is equivalent to a free-fall, we use the displacement as H and acceleration as g to get the equation $H=\frac{1}{2}gt^2$. Isolating the equation for t, $t=\sqrt{\frac{2H}{g}}$. Using this expression for time in the horizontal component, $V_j=\frac{L}{t}=\frac{L}{\sqrt{\frac{2H}{g}}}$. Isolating the equation for L, $L=V_j\sqrt{\frac{2H}{g}}$.

Substituting equation 2.1,

$$L = \sqrt{2gh} * \sqrt{\frac{2H}{g}} = \sqrt{4Hh} = 2\sqrt{Hh} \qquad - 7.1$$

The theory dealing with ideal fluids predicts that L^2 will be directly proportional to *h* by squaring equation 7.1.

7.2. Empirical data and analysis

Knowing L from *Table 1*, we can calculate L^2 for each h. *Table 2* shows the values for L^2 and uncertainties for L^2 (refer to *Appendix 4* for the full data table). Calculations are also explained below.

Table 2: Sample of raw data with processed data of average L and L^2, and uncertainty of L^2 (ΔL^2)

Trials	h (cm) ±0.1 cm	L (cm) ±0.1 cm	L^2 (cm^2)	Average L (cm)	Average L^2 (cm^2)	ΔL^2 (cm^2)
1	14.0	12.7	161.3	12.8	164	3
2		12.8	163.8			
3		12.9	166.4			
1	13.7	12.5	156.3	12.6	159	3
2		12.6	158.8			
3		12.7	161.3			
1	13.2	12.3	151.3	12.4	154	3
2		12.4	153.8			
3		12.5	156.3			
1	12.7	11.9	141.6	12.0	144	2
2		11.9	141.6			
3		12.2	148.8			

Calculating Average values

$$Average\ L\ (or\ L^2)\ = \frac{Trial\ 1 + Trial\ 2 + Trial\ 3}{3}$$

Sample calculation for h=13.7cm

$$Average\ L = \frac{12.5 + 12.6 + 12.7}{3} = 12.6\ (cm)$$

Calculating Uncertainty of L² (ΔL²)

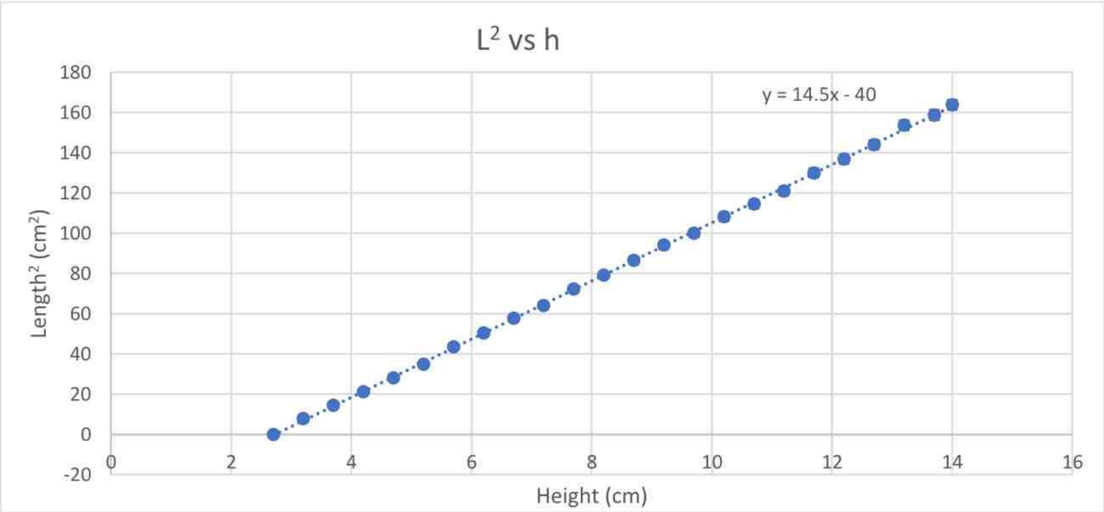
$$\Delta L^2 = percent\ uncertainty\ of\ L * 2 * L^2$$

Sample calculation for h= 13.7cm

$$\Delta L^2 = \frac{0.1}{12.6} * 2 * 159 = 3\ (cm^2)$$

Below in *Graph 1*, a graph between L^2 and h is given with a best-fit line.

Graph 1: Data points of L^2 against h



Although error bars are displayed on the graph, their magnitudes are relatively small to the scale of the axis and difficult to observe.

However, the point of interest in this graph is that unlike Torricelli's Law's prediction, the two variables are not directly proportional. Instead the line is in the form of y=k(x-q). This *q* value exists due to the energy losses arising from surface forces. The x-intercept of 2.7cm represents the height at which the pressure of water at the hole and surface forces balance, preventing the water from escaping. This *q* value also agrees with the observation where the jet of water stopped at the height of 2.7 cm. This shows the theory needs modification by using *h*-*q*, an "Effective Height" (with the notation h_F), instead of just *h*. The effective height will be the height using 2.7cm as the reference rather than the datum, as it is a better representation of a height that applies enough pressure for water to escape via the hole. Hence, Torricelli's law will be modified as

$$V_j = \sqrt{2g(h-q)} \qquad - 7.1$$

It is also worth comparing the best-fit line with the theory by the slope. The theory predicts in equation 7.1 that squaring both sides will give $L^2 = 4Hh$. The slope of this graph should be 4*H, which is equivalent to 4*3.6=14.4. This value is close but not exactly equivalent with the slope of the empirical data being 14.5. The possible reasons for this inconsistency will be further explained in the evaluation.

8.Determining the Time to Drain Tank

This section investigates the total amount of time it took for the water at initial height (h_0= 14cm) to drain completely (defined as D_0). There is also empirical data for the actual time it took for the tank to drain. However it was observed that as the height of water approached *q,* small intermittent bursts were imposed on the steady flow of water, causing V_j to fluctuate randomly and ultimately affecting D_0. This suggests that the recorded time is inaccurate, and extrapolation of accurate data points was needed to acquire the correct time the tank should have taken to drain with a steady flow. Ideally, even this factor of non-steady flows should be included in modifying the theory, but the analysis and models of such flows are too complicated even at research level. Hence, I was reduced to analyse a situation only with steady flow.

8.1. Relationship between Velocity of jet at hole (V_j) and Time (t)

From equation 7.1, an expression for h can be found by isolating *h* in equation 7.1 as

$$h - q = \frac{V_j^2}{2g} \qquad - 8.1$$

and then differentiating both sides by time gives us

$$\frac{dh}{dt} - 0 = \frac{V_j}{g} * \frac{dV_j}{dt} \qquad - 8.2$$

Referring to the continuity equation 6.2, and by substituting equation 8.2,

$$-\frac{r^2 * V_j}{R^2} =* \frac{V_j}{g} * \frac{dV_j}{dt} \qquad - 8.3$$

Isolating dV_j/dt gives

$$\frac{dV_j}{dt} = -\frac{r^2}{R^2} g \qquad - 8.4$$

Equation 8.4 shows that the rate of change of velocity over the rate of change over time should be a constant negative value. This should be visible as the slope of a line in a velocity-time graph.

8.2. Finding the Empirical Drain Time

Below, *Table 3* (refer to *Appendix 5* for full table) is a sample data table including V_j (the modified form in equation 7.1) and time (t), with the average time and the uncertainty of that value. Each calculation is explained with examples.

Table 3: Sample data of h, t, V_j, average t, and uncertainty of t

Trials	h (cm) ±0.1(cm)	t (sec)	V_j (cm/s) ±0.2(cm/sec)	Average t (sec)	Δt (sec)
1	14.0	0.0	14.9	0	0.0
2		0.0			
3		0.0			
1	13.7	7.1	14.7	8.3	0.7
2		6.1			

3		5.6			
1		17.1			
2	13.2	16.4	14.4	18.8	0.7
3		15.6			
1		28.2			
2	12.7	27.7	14.0	30.2	1.8
3		24.7			

Calculating V_j

According to equation 7.1, $V_j = \sqrt{2g(h-q)} = \sqrt{2 * 9.81(h-2.7)}$

Sample Calculation for h=13.7cm

$V_j = \sqrt{2 * 9.81(13.7-2.7)} = \sqrt{215.82} = 14.7$cm/sec

Uncertainty of V_j

Δh=Δq= ±0.1cm (ruler). Δ(h-q) = ±0.2 cm

Calculating Average time

(See calculations for Table 2)

Calculating Uncertainty of t

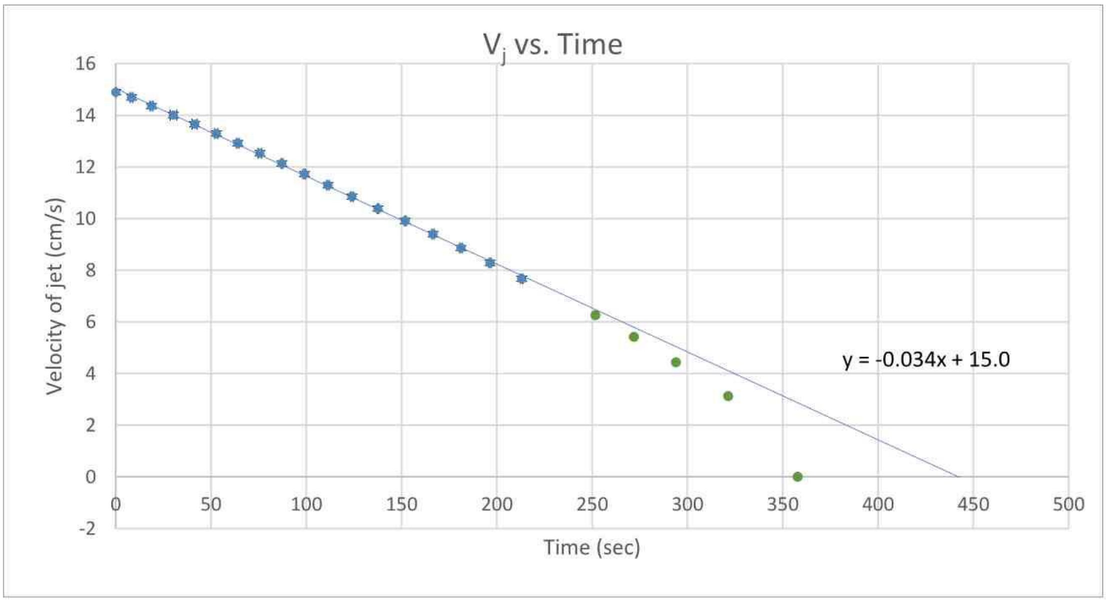
Due to many human uncertainties, the instrument's uncertainty is insignificant. Hence the range is taken from the 3 trials, $\frac{Max\ time - Min\ time}{2}$

Sample Calculation for h=13.7cm

$$\frac{7.1-5.6}{2} = 0.7sec\ (1\ significant\ figure\ as\ my\ reaction\ speed\ is\ \pm 0.1sec)$$

A graph representing the data points, their uncertainties, and a best fit line for all points except the last 5 is given below in *Graph 2*.

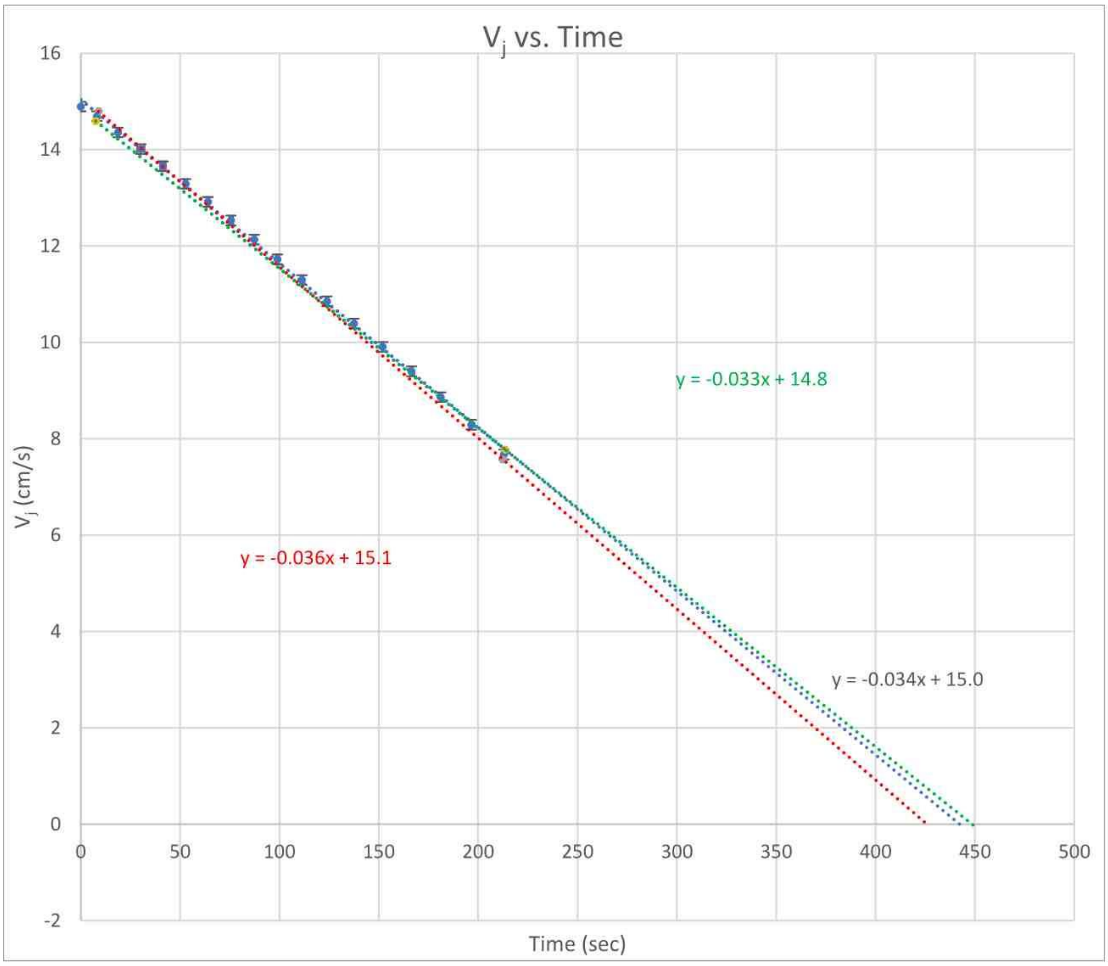
Graph 2: Data points of Velocity against Average time passed



The uncertainties are displayed here but are relatively too small to the scale of the graph and is difficult to observe.

Looking at the trend of the points, the last 5 points significantly deviate from the best fit line made by the other points. Hence it is deducible that the intermittent bursts and unstable flows observed in **Section 8** starts to noticeably affect the flow during the last 5 data points. Therefore, it is sensible to ignore the last 5 points when plotting a best fit line as the goal is to create a model for a steady flowing tank drainage. Equation 8.4 predicts the relationship between velocity of the jet and time as a straight line. The extrapolation of the best-fit line gives the x-intercept, the expected drain time (D) if the flow had stayed steady. The maximum gradient and minimum gradient of the best-fit line due to uncertainties is shown below in *Graph 3.*

Graph 3: Steepest and least steep best fit lines for V_j against time graph



Considering the uncertainties, the green line is the minimum slope the data could have whereas the red line is the line of maximum slope. The blue line is the best-fit line for the data collected as in Graph 2. It is worth noting that the theoretical prediction for the slope of the best-fit line was 0.0331 given by equation 8.4, which is within the range of the slopes, confirming the validity of the theory.

The x-intercepts of the most and least steep lines is the range of D_0 at the initial height (h_0=14cm). This value is (450-423) = 26 sec. Hence this also determines the uncertainty of the best-fit line's x-intercept, 442. The uncertainty (ΔD_0) is range/2 = 26/2 = 13 sec. Hence, D_0 can be written as 442±13 sec.

8.3. Theoretical Prediction for Drain time (D)

This theoretical proof is an adaptation of the work in LearnChemE's video.

Referring to the continuity equation 6.2 and using equation 7.1 for V_j gives

$$\frac{dh}{dt} = -\sqrt{2g} * \frac{r^2}{R^2} * \sqrt{h-q}$$

for simplicity we define the constant value $-\sqrt{2g} * \frac{r^2}{R^2}$ as *K*.

then the equation becomes $\frac{dh}{dt} = K\sqrt{h-q}$

and solving the differential equation by separating variables as

$$\frac{1}{\sqrt{h-q}}\,dh = k\,dt$$

And integrate for parameters during the first draining, hence h being from h_0 to h= q, and t from 0 to D_0

$$\int_{h_0}^{q} \frac{1}{\sqrt{h-q}}\,dh = K\int_0^{D_0} dt$$

Which gives

$$2\sqrt{h_0 - q} - 2\sqrt{q-q} = KD_0$$

Isolating D_0 gives

$$D_0 = \frac{2}{K}\sqrt{h_0 - q}$$

Substituting *K* back in gives the final equation,

$$D_0 = \frac{R^2}{r^2}\sqrt{\frac{2(h_0 - q)}{g}} \qquad - 8.4$$

Plugging in values for h_0=14cm, q=2.7, g=9.81, R=4.3, and r =0.25 gives the value of 449 seconds.

This theoretical model for h_0 is therefore validated as it is within the range of the empirical range found in **Section 8.2**.

9.Relationship between D and h_E

In equation 8.4 the variables D_0 and h_0 were used, but this formula is also a general one for any height. Hence, we can see that after squaring both sides, the squared value of the draining time should be directly proportional to the effective height h_F (=h-q). In this section we will see if the range of possible best-fit lines includes one passing the origin to confirm a directly proportional relationship.

The empirical data concerning D_0 was found in the previous **Section 8.3**. But what we want is to prove that the model for getting D works at all general heights, not just at h_0. The theoretical values for D can be found by replacing D_0 for D_{theo} and h_0 for h-q in equation 8.4, as

$$D_{theo} = \frac{R^2}{r^2}\sqrt{\frac{2(h-q)}{g}} \qquad -9.1$$

The empirical drain time can be calculated by subtracting the time it took for the water at h_0 to reach another height from the D_0 at h=14 cm. This manipulation is basically equivalent to redefining the initial height to another height than h_0, as if the experiment started at a different height. This can be written as

$$D_h = D_0 - t \qquad -9.2$$

Where D_h is the empirical drain time at a height *h* and t is the time taken by the water to reach a certain height, i.e. the average t value.

A sample of the data including the necessary calculations (including uncertainties) is given below in *Table 4* (refer to *Appendix 6* for full table)

Table 4: Sample data table with h_F, t, D_h, D_{theo}, $(D_h)^2$, $(D_{theo})^2$, the uncertainty of D (ΔD), and $(\Delta D)^2$

h_F (cm) ±0.05	Average t (sec)	D_h (sec)	D_{theo} (sec)	$(D_h)^2$ (sec^2)	$(D_{theo})^2$ (sec^2)	ΔD_h (sec)	$\Delta(D_h)^2$ (sec^2)
11.3	0	442	449	1.96E+05	201629	13.2	1.16E+04
11.0	8.3	434	443	1.88E+05	196276	13.9	1.20E+04
10.5	18.8	423	433	1.79E+05	187354	13.9	1.17E+04
10.0	18.8	412	422	1.70E+05	178433	14.9	1.23E+04
9.5	30.2	401	412	1.61E+05	169511	14.5	1.16E+04

Calculating Average D_h

Using equation 9.2, substitute values

Sample Calculation for D_h at h_F=10.5cm

$$D_h = 442 - 18.77 = 423\ sec$$

Calculating D_{theo}

Use equation 9.1 and substitute values

Sample Calculations for D_{theo} for h_F=11.0cm

$$D_{theo} = \frac{4.3^2}{0.25^2}\sqrt{\frac{2 * 11.0}{9.81}} = 443\ sec$$

Calculating Uncertainty of D_h

Since $D_h = D_0 - t$, $\Delta D_h = \Delta t + \Delta D_0$

Sample Calculation for ΔD_h at h_F=10.5cm

Δt at h_F=10.5cm is ±0.7sec. Adding this to the initial ΔD_h of ±13.2 gives a significant value of ±13.9

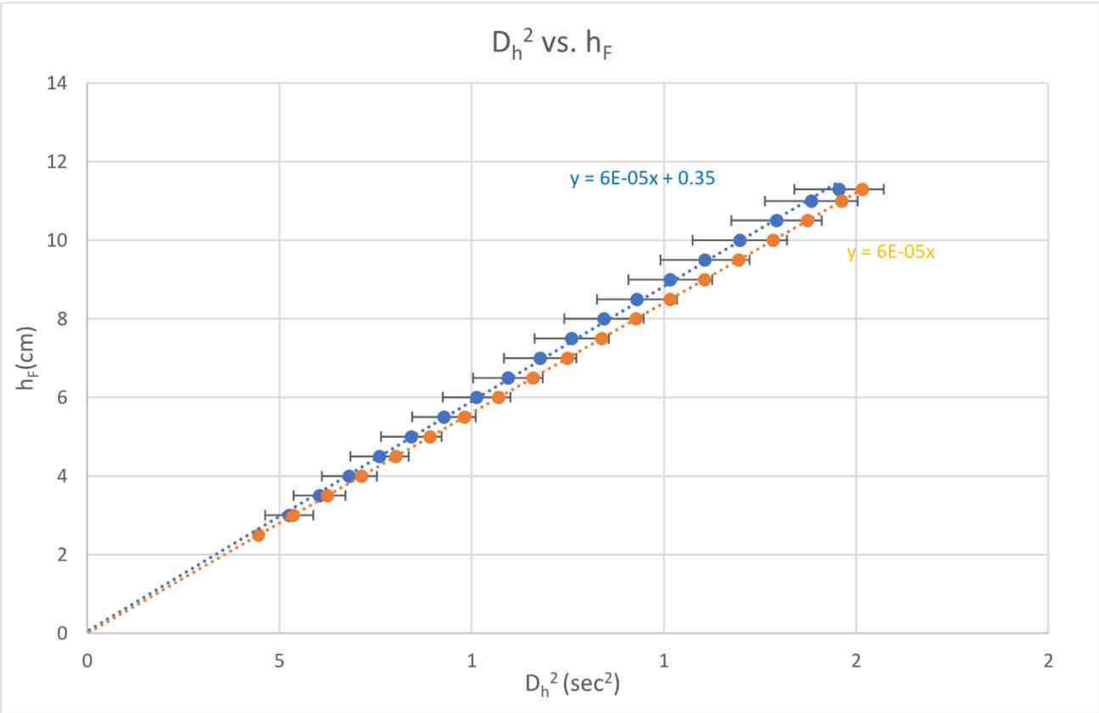
Calculating Uncertainty of $(D_h)^2$

Same method is used as in calculating uncertainty of L^2 (refer to calculations for *Table 2*)

The data represented here have taken significant figures into account, limiting the values to reasonable amounts of significant figures. For example, the value for D_h^2 was limited to 3 significant figures as the D_h value was limited to 3 significant figures.

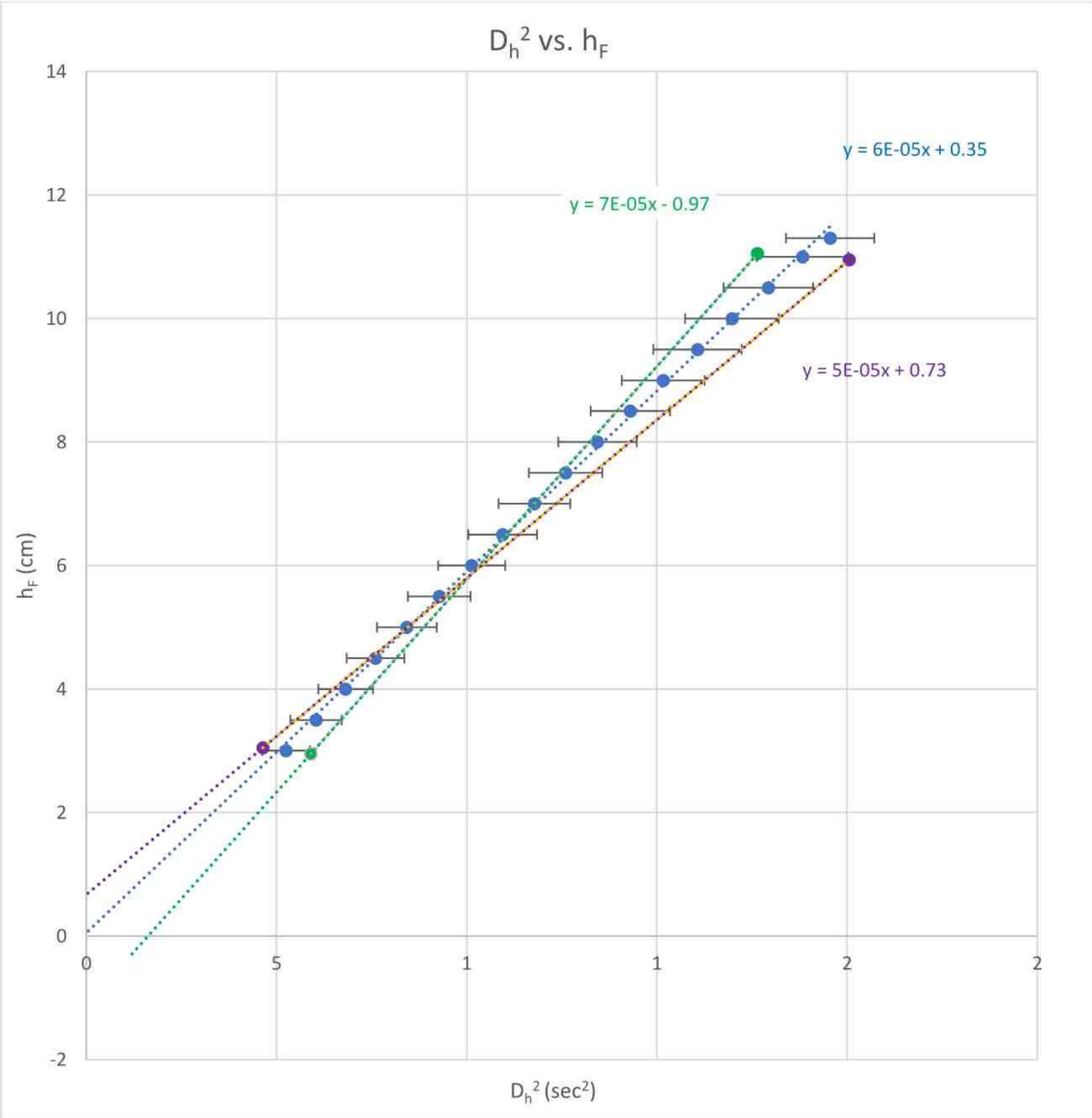
To find a relationship between D_h^2 and h_F, below is *Graph 4*, giving the best-fit line for the 2 variables, also with D_{Theo} data points for a visual comparison.

Graph 4: Data points for D_h and D_{Theo} against h_F



As expected from equation 9.1, the theoretical line (Orange) passes through the origin, confirming the directly proportional relationship. Also, the same 5 points of data ignored in *Graph 2* were also excluded in this graph as the intermittent bursts distort the steady flow and distort the time taken for draining. It is also because the intermittent bursts make the jet unstable and very difficult to get precise measurements of L and flow at random speeds irrelevant to the pressure of water. It is also important to note that the empirical gradient has the same slope as the theoretical prediction, suggesting the relationship could be valid but was disturbed by errors. The error bars are representations of these uncertainties, suggesting that a steepest and least steep line could be calculated to see if the empirical data corresponds with the theoretical prediction. This range of slopes is shown below as *Graph 5.*

Graph 5: Range of possible best-fit lines for D_h^2 against h_F



As the first point of measurement has more uncertainty than other points as human errors (reaction speed, determining when to start measurements, etc.), the range of slopes is taken with the last and 2'nd points. The green and purple lines are respectively the steepest and least steep lines. We can see that the range of possible best-fit lines include the origin, just like the theory predicted. This is convincing evidence supporting the theory and the overall method used in the experiment and analysis.

10.Conclusion and Evaluation

This process of modifying Torricelli's law and deriving new knowledge from an original theory has led to several interesting conclusions.

First, we observed that the jet of water would stop before the water in the tank reached the hole, and that this height remained constant for all trials. This lead to the first modification to the theory, replacing the height of the water with an effective height that takes energy losses into account.

Secondly, we noticed that as the velocity of the jet decreased with time, the jet of water started to destabilise and fluctuate with intermittent bursts. This effect got more noticeable as time passed and the height decreased, giving a distorted velocity and eventually a misled drain time. The effect was also clear from the increasing deviations of the last 5~6 points from the predicted straight line. This investigation was limited to only deal with steady flows, thus forced me to find the hypothetical drain time if the flow had been consistently steady by extrapolating the velocity and time graph. We also found the range of the possible drain times through plotting the range of the gradients using uncertainty values.

Finally, after making these modifications and assumptions we could create and verify a model to describe the relationship between D_h and h_F, which we found to be a directly proportional relationship. Considering the uncertainties in the measured values, the range of the possible best-fit lines had one that passed through the origin, supporting our theory to be valid.

Overall, we can conclude that we better understand the phenomenon of draining a tank with our newly refined model. After all the modifications we created a new model and gained new knowledge from it that the drain time is directly proportional to the effective height. Now knowing this, we can explain this as the effective height being an accurate representation the balanced pressure that can create the jet of water. Hence the amount of time to drain a tank depends directly on the effective height of the water. However, there are still many problems and uncertainties in this investigation that need to be noted.

Part of the uncertainties arise from the limitations the setup has. For example, the camera taking measurements for L was fixed in one place, meaning most measurements were taken at a diagonal angle. This would make the jet appear to be at a different length on the screen than it actually is, longer at first and shorter at end (also seen in the video when the jet appears to be at 0 on the ruler, but still has some velocity). However as seen in *Graph 1,* this error is small enough to give reasonable data, although it causes the empirical graph to have an inaccurate slope. This problem with level measurement is also the reason the height of the water was not recorded by a camera unlike my initial plan as the discrepancies were too severe and I had to resort to use my own eyes to follow the level of the water. The problem with changing to this method is that I could not gather multiple data points for height against time as it was roughly done with my eyes, reaction speed, and the decision about when to start measuring. This would result in fixing the initial data point to always be 14cm and 0 seconds, which is unlikely to be true. Also, as visible in *Picture 1*, the jet of water was not a perfect laminar flow at the level of the ruler (because the water was not clear/see-through and hazy) and had minor fluctuations, moving constantly up and down. This random error would give inaccurate measurements for L, the most likely reason for the inaccurate slope in **Section 7.2**.

Another limitation is that there was not any way of directly measuring V_j and verifying Torricelli's law. We had to use indirect verifications involving lengths and time, thus further increasing the uncertainties.

Quite a few questions emerge from this investigation. The major unresolved problem is the exclusion of the small intermittent bursts mentioned in **Section 8** while creating our model. The goal of this investigation was to try and create a model to perfectly describe and quantify the tank draining phenomenon. However after research, an analytical approach to these bursts seemed to be interesting, but way beyond my level and too complex to be properly done.

Word Count: 3921

Works Cited

Oliveira, Paulo Murilo Castro De, et al. "Pin-Hole water flow from cylindrical bottles." *Physics Education*, vol. 35, no. 2, 2000, pp. 110–119., doi:10.1088/0031-9120/35/2/306.

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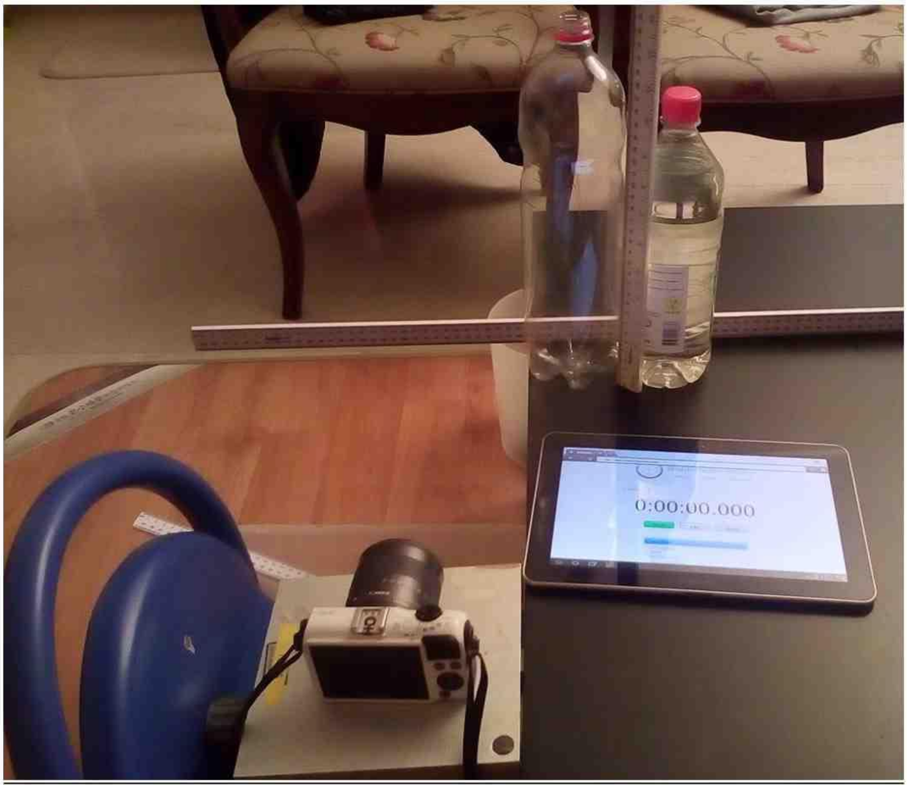
11. Appendix 1: Picture of setup



12. Appendix 2: Full table of *Table 1*

Trials	h (cm) ±0.05	time (sec)	L (cm) ±0.05
1	14.0	0.00	12.7
2		0.00	12.8
3		0.00	12.9
1	13.7	7.06	12.5
2		6.06	12.6
3		5.63	12.7
1	13.2	17.05	12.3
2		16.43	12.4
3		15.63	12.5
1	12.7	28.19	11.9
2		27.65	11.9

3		24.65	12.2
1	12.2	38.85	11.6
2		37.63	11.7
3		36.23	11.8
1	11.7	49.36	11.3
2		48.69	11.4
3		47.69	11.5
1	11.2	60.15	10.9
2		59.37	11.0
3		58.92	11.1
1	10.7	71.53	10.6
2		70.77	10.7
3		69.65	10.8
1	10.2	82.21	10.3
2		81.99	10.4
3		81.38	10.5
1	9.7	93.98	9.9
2		93.25	10.0
3		92.85	10.0
1	9.2	105.89	9.7
2		105.38	9.7
3		104.75	9.8
1	8.7	118.53	9.3
2		117.96	9.3
3		117.18	9.4
1	8.2	130.56	8.9
2		130.17	8.9
3		129.75	9.0
1	7.7	144.51	8.5
2		144.04	8.5
3		143.69	8.5
1	7.2	158.86	8.0
2		158.14	8.0
3		157.72	8.0
1	6.7	173.12	7.6
2		172.38	7.6
3		171.97	7.6
1	6.2	187.87	7.1
2		187.50	7.1
3		186.79	7.1
1	5.7	204.13	6.6
2		203.68	6.6
3		202.96	6.6

1	5.2	221.43	5.9
2		221.50	5.9
3		220.60	5.9
1	4.7	239.81	5.3
2		239.31	5.3
3		238.99	5.3
1	4.2	259.51	4.6
2		259.12	4.6
3		258.64	4.6
1	3.7	281.35	3.8
2		281.00	3.8
3		280.14	3.8
1	3.2	308.85	2.8
2		307.87	2.8
3		307.39	2.8
1	2.7	347.82	0.0
2		347.08	0.0
3		345.83	0.0

13. Appendix 3: Calculations for Torricelli's law deviation

Below is *equation 6.6,* the correct version of Torricelli's law derived from Bernoulli's equation. The calculation of its deviation to the conventional Torricelli's law is presented here.

$$V_j^2\left(1-\frac{r^4}{R^4}\right)=2gh \qquad -6.6$$

$$V_j^2=\frac{2gh}{1-\frac{r^4}{R^4}}$$

$$V_j=\sqrt{\frac{2gh}{1-\frac{r^4}{R^4}}}$$

In this calculation we will use the case for when h_F=11.3 (as we are at the end of the essay it makes sense to use the modified version of Torricelli's law, refer to *equation 7.1*).

Hence, r=0.25, R=4.3, g=9.81, h_F=11.3

$$V_j = \sqrt{\frac{2 * 9.81 * 11.3}{1 - \frac{0.25^4}{4.3^4}}} = 14.8898802 \ldots$$

Comparing this to the V_j that we would have gotten using the conventional Torricelli's law, 14.88980, we can see that the deviation is in the magnitude of 10^{-5}.

14. Appendix 4: Full data table for *Table 2*

Trials	h (cm) ±0.05	L (cm) ±0.05	L^2 (cm^2)	Average L (cm)	Average L^2 (cm^2)	ΔL^2
1	14.0	12.7	161.3	12.8	163.8	2.6
2		12.8	163.8			
3		12.9	166.4			
1	13.7	12.5	156.3	12.6	158.8	2.5
2		12.6	158.8			
3		12.7	161.3			
1	13.2	12.3	151.3	12.4	153.8	2.5
2		12.4	153.8			
3		12.5	156.3			
1	12.7	11.9	141.6	12	144.0	3.6
2		11.9	141.6			
3		12.2	148.8			
1	12.2	11.6	134.6	11.7	136.9	2.3
2		11.7	136.9			
3		11.8	139.2			
1	11.7	11.3	127.7	11.4	130.0	2.3
2		11.4	130.0			
3		11.5	132.3			
1	11.2	10.9	118.8	11	121.0	2.2
2		11.0	121.0			
3		11.1	123.2			
1	10.7	10.6	112.4	10.7	114.5	2.1
2		10.7	114.5			
3		10.8	116.6			
1	10.2	10.3	106.1	10.4	108.2	2.1
2		10.4	108.2			
3		10.5	110.3			

1	9.7	9.9	98.0	10	100.0	1.0
2		10.0	100.0			
3		10.0	100.0			
1	9.2	9.7	94.1	9.7	94.1	1.0
2		9.7	94.1			
3		9.8	96.0			
1	8.7	9.3	86.5	9.3	86.5	0.9
2		9.3	86.5			
3		9.4	88.4			
1	8.2	8.9	79.2	8.9	79.2	0.9
2		8.9	79.2			
3		9.0	81.0			
1	7.7	8.5	72.3	8.5	72.3	0.0
2		8.5	72.3			
3		8.5	72.3			
1	7.2	8.0	64.0	8	64.0	0.0
2		8.0	64.0			
3		8.0	64.0			
1	6.7	7.6	57.8	7.6	57.8	0.0
2		7.6	57.8			
3		7.6	57.8			
1	6.2	7.1	50.4	7.1	50.4	0.0
2		7.1	50.4			
3		7.1	50.4			
1	5.7	6.6	43.6	6.6	43.6	0.0
2		6.6	43.6			
3		6.6	43.6			
1	5.2	5.9	34.8	5.9	34.8	0.0
2		5.9	34.8			
3		5.9	34.8			
1	4.7	5.3	28.1	5.3	28.1	0.0
2		5.3	28.1			
3		5.3	28.1			
1	4.2	4.6	21.2	4.6	21.2	0.0
2		4.6	21.2			
3		4.6	21.2			
1	3.7	3.8	14.4	3.8	14.4	0.0
2		3.8	14.4			
3		3.8	14.4			
1	3.2	2.8	7.8	2.8	7.8	0.0
2		2.8	7.8			
3		2.8	7.8			
1	2.7	0.0	0.0	0	0.0	

2		0.0	0.0			0.0
3		0.0	0.0			

15. Appendix 5: Full data table for *Table 3*

Trials	h (cm) ±0.05	t (sec)	V_j (cm/s) ±0.1	Average t (sec)	Δt
1	14.0	0.00	14.9	0	0.00
2		0.00			
3		0.00			
1	13.7	7.06	14.7	8.25	0.72
2		6.06			
3		5.63			
1	13.2	17.05	14.4	18.77	0.71
2		16.43			
3		15.63			
1	12.7	28.19	14.0	30.23	1.77
2		27.65			
3		24.65			
1	12.2	38.85	13.7	41.37	1.31
2		37.63			
3		36.23			
1	11.7	49.36	13.3	52.78	0.84
2		48.69			
3		47.69			
1	11.2	60.15	12.9	64.08	0.61
2		59.37			
3		58.92			
1	10.7	71.53	12.5	75.65	0.94
2		70.77			
3		69.65			
1	10.2	82.21	12.1	87.26	0.41
2		81.99			
3		81.38			
1	9.7	93.98	11.7	99.02	0.57
2		93.25			
3		92.85			
1	9.2	105.89	11.3	111.4	0.57
2		105.38			
3		104.75			

1	8.7	118.53	10.8	123.95	0.67
2		117.96			
3		117.18			
1	8.2	130.56	10.4	137.62	0.41
2		130.17			
3		129.75			
1	7.7	144.51	9.9	151.94	0.41
2		144.04			
3		143.69			
1	7.2	158.86	9.4	166.5	0.57
2		158.14			
3		157.72			
1	6.7	173.12	8.9	181.15	0.58
2		172.38			
3		171.97			
1	6.2	187.87	8.3	196.48	0.54
2		187.50			
3		186.79			
1	5.7	204.13	7.7	213.05	0.58
2		203.68			
3		202.96			
1	5.2	221.43	7.0	232.02	0.45
2		221.50			
3		220.60			
1	4.7	239.81	6.3	251.73	0.41
2		239.31			
3		238.99			
1	4.2	259.51	5.4	271.85	0.44
2		259.12			
3		258.64			
1	3.7	281.35	4.4	293.99	0.61
2		281.00			
3		280.14			
1	3.2	308.85	3.1	321.43	0.73
2		307.87			
3		307.39			
1	2.7	347.82	0.0	357.87	1.00
2		347.08			
3		345.83			

16. Appendix 6: Full data table for *Table 4*

Average t (sec)	Average D_h (sec)	h_F(cm) ±0.05	D_{Theo} (sec)	Average D_h^2 (sec^2)	D_{Theo}^2 (sec^2)	ΔD_h	$\Delta (D_h)^2$
0	442	11.3	449	1.96E+05	201629	13.2	1.16E+04
8.25	434	11.0	443	1.88E+05	196276	13.9	1.20E+04
18.77	423	10.5	433	1.79E+05	187354	13.9	1.17E+04
30.23	412	10.0	422	1.70E+05	178433	14.9	1.23E+04
41.37	401	9.5	412	1.61E+05	169511	14.5	1.16E+04
52.78	389	9.0	401	1.52E+05	160590	14.0	1.09E+04
64.08	378	8.5	389	1.43E+05	151668	13.8	1.04E+04
75.65	367	8.0	378	1.34E+05	142746	14.1	1.03E+04
87.26	355	7.5	366	1.26E+05	133825	13.6	9.63E+03

99.02	343	7.0	353	1.18E+05	124903	13.7	9.41E+03
111.4	331	6.5	341	1.09E+05	115981	13.7	9.08E+03
123.95	318	6.0	327	1.01E+05	107060	13.8	8.80E+03
137.62	305	5.5	313	9.28E+04	98138	13.6	8.26E+03
151.94	290	5.0	299	8.43E+04	89216	13.6	7.87E+03
166.5	276	4.5	283	7.60E+04	80295	13.7	7.57E+03
181.15	261	4.0	267	6.82E+04	71373	13.7	7.17E+03
196.48	246	3.5	250	6.04E+04	62451	13.7	6.73E+03
213.05	229	3.0	231	5.25E+04	53530	13.7	6.30E+03
232.02	210	2.5	211	4.42E+04	44608	13.6	5.72E+03

251.73		2.0				13.6	
271.85		1.5				13.6	
293.99		1.0				13.8	
321.43		0.5				13.9	
357.87		0.0				14.1	